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# CHARGE CONJUGATION IN THE GALILEAN LIMIT

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*Strictly working in the framework of the nonrelativistic quantum mechanics of a spin $\frac{1}{2}$ particle coupled to an external electromagnetic field, we show, by explicit construction, the existence of a charge conjugation operator matrix which defines the corresponding antiparticle wave function and leads to the galilean and gauge invariant Schroedinger-Pauli equation satisfied by it.*

Key words: charge conjugation; galilean relativity; gauge invariance.

## 1. **Introduction**

In a recent paper [1], Cabo *et al* showed the *existence* of the nonrelativistic limit $C_{nr}$ of the charge conjugation operation $C$ for the Dirac equation of a 4-spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ coupled to an external electromagnetic potential $(\phi, \vec{A})$. At low velocities of the Dirac particle with respect to the velocity of light in vacuum $c$, the "large components" $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of $\Psi$ satisfy the Schroedinger-Pauli equation [2]

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2m}(-\nabla^2 + \frac{q^2}{\hbar^2 c^2}\vec{A}^2 + \frac{iq}{\hbar c}\nabla \cdot \vec{A} + \frac{2iq}{\hbar c}\vec{A} \cdot \nabla - \frac{q}{\hbar c}\vec{\sigma} \cdot \vec{B} + 2mq\phi) \begin{pmatrix} u \\ v \end{pmatrix} \tag{1}$$

where $q$ and $m$ are the electric charge and mass respectively, $\vec{\sigma}$ are the Pauli matrices, $\vec{B} = \nabla \times \vec{A}$ is the magnetic field and, at each space time point $\begin{pmatrix} \vec{x} \\ t \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2$. The *charge conjugate* Pauli spinor $\psi_c$ representing spin $\frac{1}{2}$ antiparticles (e.g. positrons) if $\psi$ represents spin $\frac{1}{2}$ particles (e.g. electrons) is given by

$$\psi_c = \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = C_{nr} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

where

$$C_{nr} = KM, \quad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

is the nonrelativistic limit of the charge conjugation matrix of the Dirac equation, which up to a sign is given by [3]

$$C = i\gamma^2\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \tag{4}$$

$K$ is the complex conjugation antilinear and hemitian $(K^\dagger = K)$ operation. $\psi_c$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} = \frac{1}{2m}(\nabla^2 - \frac{q^2}{\hbar^2 c^2}\vec{A}^2 + \frac{iq}{\hbar c}\nabla \cdot \vec{A} + \frac{2iq}{\hbar c}\vec{A} \cdot \nabla - \frac{q}{\hbar c}\vec{\sigma} \cdot \vec{B} - 2mq\phi) \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix}. \tag{5}$$

As it was proved in reference 1, both (1) and (5) are transformed into each other by the operator $C_{nr}$, thus reaffirming the *galilean character of the approximation $C_{nr}$ to $C$*. This is a non trivial result specially because of the general belief that charge conjugation is a symmetry that exists only in the relativistic regime. [4]

In this note we discuss the previous result without appealing to the limiting process, namely, strictly working in the context of the galilean group, for simplicity of its connected component $G_0$, and of its universal covering group $\hat{G}_0$ (section 2). From the lagrangian density $\mathcal{L}$ for the equation (1), and using $C_{nr}$, we construct the lagrangian density $\mathcal{L}_c$ for equation (5), and prove the galilean invariance of these equations by proving this invariance for $\mathcal{L}$ and $\mathcal{L}_c$. We also verify the gauge invariance of $\mathcal{L}_c$ (section 3).

## 2. **Galilean group, its universal covering group, and spinors**

The connected component of the galilean group $G_0$ consists of the set of 4×4 matrices

$$g = \begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix} \tag{6}$$

with $R$ in the 3-dimensional rotation group $SO(3)$, boost velocity $\vec{V}$ in $\mathbb{R}^3$, composition law

$$g_2 g_1 = \begin{pmatrix} R_2 & \vec{V}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & \vec{V}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & \vec{V}_2 + R_2 \vec{V}_1 \\ 0 & 1 \end{pmatrix}, \tag{6$a$}$$

identity

$$\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6$b$}$$

and inverse

$$\begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}\vec{V} \\ 0 & 1 \end{pmatrix}. \tag{6$c$}$$

$G_0$ is a non abelian, non compact, connected but non simply connected six dimensional Lie group; like the connected component of the Lorentz group, its topology is that of the cartesian product of the real projective space with ordinary 3-space *i.e.* of $\mathbb{R}P^3 \times \mathbb{R}^3$. The action of $G_0$ on spacetime is given by

$$G_0 \times \mathbb{R}^4 \to \mathbb{R}^4, \; (g, \begin{pmatrix} \vec{x}' \\ t' \end{pmatrix}) \mapsto \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = g \begin{pmatrix} \vec{x}' \\ t' \end{pmatrix} = \begin{pmatrix} R\vec{x}' + \vec{V}t' \\ t' \end{pmatrix}. \tag{7}$$

Since one has the action

$$\mu : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3, \; (R, \vec{x}) \mapsto R\vec{x}, \tag{8}$$

then $G_0$ is isomorphic to the semidirect sum $\mathbb{R}^3 \times_\mu SO(3)$: $\begin{pmatrix} R & \vec{V} \\ 0 & 1 \end{pmatrix} \mapsto (\vec{V}, R)$ with composition law

$$(\vec{V}', R')(\vec{V}, R) = (\vec{V}' + R'\vec{V}, R'R). \tag{8$a$}$$

The *universal covering group* of $G_0$ is given by the $\mathbb{Z}_2$-bundle

$$\mathbb{Z}_2 \to \hat{G}_0 \xrightarrow{\Pi} G_0 \tag{9}$$

where

$$\hat{G}_0 = \{\hat{g} = \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}, \; T \in SU(2), \; \vec{V} \in \mathbb{R}^3\}, \tag{9$a$}$$

and $\Pi$ is the $2 \to 1$ group homomorphism

$$\Pi(\hat{g}) = \begin{pmatrix} \pi(T) & \vec{V} \\ 0 & 1 \end{pmatrix} \tag{9$b$}$$

with $\pi : SU(2) \to SO(3)$ the well known projection

$$\pi \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} Rez^2 - Rew^2 & Imz^2 + Imw^2 & -2Rezw \\ -Imz^2 + Imw^2 & Rez^2 + Rew^2 & 2Imzw \\ 2Rez\bar{w} & 2Imz\bar{w} & |z|^2 - |w|^2 \end{pmatrix}. \tag{9c}$$

$\hat{G}_0$ is simply connected and has the topology of $S^3 \times \mathbb{R}^3$. Since $SU(2)$ acts on $\mathbb{R}^3$:

$$\hat{\mu} : SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3, \ (T, \vec{V}) \mapsto \pi(T)\vec{V}, \tag{10}$$

one has the group isomorphism

$$\hat{G}_0 \ni \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} \mapsto (\vec{V}, T) \in \mathbb{R}^3 \times_{\hat{\mu}} SU(2); \tag{11}$$

the composition law in $\hat{G}_0$ is given by

$$\begin{pmatrix} T' & \vec{V}' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T'T & \vec{V}' + \pi(T')\vec{V} \\ 0 & 1 \end{pmatrix}, \tag{12}$$

while the identity and inverse are respectively given by

$$\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{12a}$$

and

$$\begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} T^{-1} & -\pi(T^{-1})\vec{V} \\ 0 & 1 \end{pmatrix}. \tag{12b}$$

Turning back to physics, for each mass value $m > 0$, $\hat{G}_0$ acts on the infinite dimensional Hilbert space $\mathcal{L}_1^2$ of continuously differentiable and square integrable $\mathbb{C}^2$-valued functions $\begin{pmatrix} u \\ v \end{pmatrix}$ on $\mathbb{R}^4$, the Schroedinger-Pauli spinors. This action is defined as follows: [5]

$$\hat{\mu}_m : \hat{G}_0 \times \mathcal{L}_1^2 \to \mathcal{L}_1^2, \ (\begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}) \mapsto \begin{pmatrix} T & \vec{V} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} : \mathbb{R}^4 \to \mathbb{C}^2,$$

$$\begin{pmatrix} \vec{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} T & \vec{V} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = e^{\frac{-im}{\hbar}(\vec{V}\cdot\vec{x} + \frac{1}{2}|\vec{V}|^2 t)} T \begin{pmatrix} u(\pi(T)\vec{x} + \vec{V}t, t) \\ v(\pi(T)\vec{x} + \vec{V}t, t) \end{pmatrix}. \tag{13}$$

$\hat{\mu}_m$ is equivalent to the *representation*

$$\tilde{\hat{\mu}}_m : \hat{G}_0 \to End(\mathcal{L}_1^2), \ \tilde{\hat{\mu}}_m(\hat{g})(\begin{pmatrix} u \\ v \end{pmatrix}) = \hat{g} \cdot \begin{pmatrix} u \\ v \end{pmatrix}. \tag{13.a}$$

At each $t$ one has the inner product

$$(\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix})(t) = \int d^3\vec{x}(\bar{u}_2(\vec{x}, t)u_1(\vec{x}, t) + \bar{v}_2(\vec{x}, t)v_1(\vec{x}, t)) \tag{14a}$$

and the norm

$$|| \begin{pmatrix} u \\ v \end{pmatrix} ||^2(t) = (\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix})(t) = \int d^3\vec{x}(|u(\vec{x}, t)|^2 + |v(\vec{x}, t)|^2). \tag{14b}$$

The galilean transformation of the charge conjugate spinor $\psi_c$ is given by

$$\psi_c \mapsto \bar{\hat{g}} \cdot \psi_c, \ \begin{pmatrix} \bar{T} & \vec{V} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \end{pmatrix} = e^{\frac{im}{\hbar}(\vec{V}\cdot\vec{x} + \frac{1}{2}|\vec{V}|^2 t)} \bar{T} \begin{pmatrix} -\bar{v}(\pi(\bar{T})\vec{x} + \vec{V}t, t) \\ \bar{u}(\pi(\bar{T})\vec{x} + \vec{V}t, t) \end{pmatrix}. \tag{15}$$

Finally, the galilean transformations of the electromagnetic potential $(\phi,\vec{A})$ and the magnetic field $\vec{B}$ are

$$\phi(\vec{x},t)=\phi'(\vec{x}',t'),\ \ \vec{A}(\vec{x},t)=R\vec{A}'(\vec{x}',t'),\ \ \vec{B}(\vec{x},t)=R\vec{B}'(\vec{x}',t') \tag{16}$$

with $\vec{x}=R\vec{x}'+\vec{V}t'$ and $t=t'$.

*Remark*: Representations associated with different values of the mass are inequivalent. [6]

3. **Lagrangian formulation and galilean and gauge invariances**

The Pauli equations (1) and (5) can be formulated within the lagrangian framework. The lagrangian for equation (1) is

$$\mathcal{L}=\frac{i\hbar}{2}(((\frac{\partial}{\partial t}-\frac{iq}{\hbar}\phi)\psi^{\dagger})\psi-\psi^{\dagger}(\frac{\partial}{\partial t}+\frac{iq}{\hbar}\phi)\psi)+\frac{\hbar^2}{2m}(\nabla+\frac{iq}{\hbar c}\vec{A})\psi^{\dagger}\cdot(\nabla-\frac{iq}{\hbar c}\vec{A})\psi-\frac{q\hbar}{2mc}\psi^{\dagger}\vec{\sigma}\cdot\vec{B}\psi$$

$$=\frac{i\hbar}{2}(\dot{\psi}^{\dagger}\psi-\psi^{\dagger}\dot{\psi})+\frac{\hbar^2}{2m}\nabla\psi^{\dagger}\cdot\nabla\psi+\frac{q^2}{2mc^2}\psi^{\dagger}|\vec{A}|^2\psi+\frac{i\hbar q}{2mc}(\psi^{\dagger}\vec{A}\cdot\nabla\psi-\nabla\psi^{\dagger}\cdot\vec{A}\psi)-\frac{q\hbar}{2mc}\psi^{\dagger}\vec{\sigma}\cdot\vec{B}\psi+q\psi^{\dagger}\phi\psi, \tag{17}$$

and equation (1) amounts to the variational equation

$$\frac{\delta}{\delta\psi^{\dagger}(\vec{x},t)}S=0 \tag{18}$$

where $S$ is the action

$$S=\int dt\int d^3\vec{x}\mathcal{L}(\vec{x},t). \tag{19}$$

Under the charge conjugation operation

$$\mathcal{L}\to\mathcal{L}_c=K\mathcal{L}=-\frac{i\hbar}{2}(((\frac{\partial}{\partial t}+\frac{iq}{\hbar}\phi)\psi_c^{\dagger})\psi_c-\psi_c^{\dagger}(\frac{\partial}{\partial t}-\frac{iq}{\hbar}\phi)\psi_c)+\frac{\hbar^2}{2m}(\nabla-\frac{iq}{\hbar c}\vec{A})\psi_c^{\dagger}\cdot(\nabla+\frac{iq}{\hbar c}\vec{A})\psi_c+\frac{q\hbar}{2mc}\psi_c^{\dagger}\vec{\sigma}\cdot\vec{B}\psi_c$$

$$=-\frac{i\hbar}{2}(\dot{\psi}_c^{\dagger}\psi_c-\psi_c^{\dagger}\dot{\psi}_c)+\frac{\hbar^2}{2m}\nabla\psi_c^{\dagger}\cdot\nabla\psi_c+\frac{q^2}{2mc^2}\psi_c^{\dagger}|\vec{A}|^2\psi_c-\frac{i\hbar q}{2mc}(\psi_c^{\dagger}\vec{A}\cdot\nabla\psi_c-\nabla\psi_c^{\dagger}\cdot\vec{A}\psi_c)+\frac{q\hbar}{2mc}\psi_c^{\dagger}\vec{\sigma}\cdot\vec{B}\psi_c+q\psi_c^{\dagger}\phi\psi_c. \tag{20}$$

To pass from (17) to (20), the identity $M^{\dagger}M=1$ is inserted at each term of (17), and the fact that $M\vec{\sigma}M^{\dagger}=M(\sigma_1,\sigma_2,\sigma_3)M^{\dagger}=(-\sigma_1,\sigma_2,-\sigma_3)$ is used; then the complex conjugation operation $K$ completes the transformation.

The total action for the particle-antiparticle system is

$$S_{tot}=S+S_c=\int dt\int d^3\vec{x}(\mathcal{L}(\vec{x},t)+\mathcal{L}_c(\vec{x},t)) \tag{21}$$

and equation (5) is obtained from $S_{tot}$ or $S_c$ as

$$\frac{\delta}{\delta\psi_c^{\dagger}(\vec{x},t)}S_{tot}=\frac{\delta}{\delta\psi_c^{\dagger}(\vec{x},t)}S_c=0. \tag{22}$$

The lagrangian $\mathcal{L}$, and therefore the equation (1), are invariant under the galilean transformations (13), (15) and (16) for $\psi$, $\psi_c$, and $(\phi,\vec{A})$ and $\vec{B}$, respectively. To prove it, we use the facts that $\nabla=R^{-1}\nabla'$ where $\nabla=\frac{\partial}{\partial\vec{x}}$ and $\nabla'=\frac{\partial}{\partial\vec{x}'}$, and $\frac{\partial}{\partial t}=\frac{\partial}{\partial t'}-R^{-1}\vec{V}\cdot\nabla'$. If $\mathcal{L}'$ and $\mathcal{L}_c'$ are the transformed lagrangian densities for particles and antiparticles, then, from equation (20),

$$\mathcal{L}_c(\vec{x},t)=K\mathcal{L}(\vec{x},t)=K\mathcal{L}'(\vec{x}',t')=\mathcal{L}_c'(\vec{x}',t') \tag{23}$$

and therefore the galilean invariance of equation (5) is also proved.

Finally, both $\mathcal{L}$ and $\mathcal{L}_c$, and therefore the equations (1) and (5), are gauge invariant under the transformations $\psi \rightarrow e^{i\Lambda}\psi$, $\psi_c \rightarrow e^{-i\Lambda}\psi_c$, $\phi \rightarrow \phi - \frac{\hbar}{q}\frac{\partial}{\partial t}\Lambda$ and $\vec{A} \rightarrow \vec{A} + \frac{\hbar c}{q}\nabla\Lambda$, where $\Lambda$ is an arbitrary differentiable function of $(\vec{x}, t)$.

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